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Tetromino tilings and the Tutte polynomial

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Abstract
We consider tiling rectangles of size $4m \times 4n$ by T-shaped tetrominoes. Each tile is assigned a weight that depends on its orientation and position on the lattice. For a particular choice of the weights, the generating function of tilings is shown to be the evaluation of the multivariate Tutte polynomial $Z_G(Q, \mathbf{v})$ (known also to physicists as the partition function of the Q-state Potts model) on an $(m-1) \times (n-1)$ rectangle G, where the parameter Q and the edge weights $\mathbf{v}$ can take arbitrary values depending on the tile weights.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The problem of evaluating the number of ways a given planar domain can be covered with a prescribed set of tiles (without leaving any holes or overlaps) has a long history in recreational mathematics, enumerative combinatorics and theoretical physics. While such counting problems are intrinsically interesting in their own right, they also play an important role in describing the statistical properties of the adsorption of variously shaped molecules to a crystalline substrate (and as such have experimental realizations), or serve as idealized models in the physics of liquid crystals. The number of tiling configurations to be counted then plays the role of a partition function.

One important question is whether a given tiling problem leads to a macroscopically ordered state. In many situations the answer turns out to be no, and the thermodynamic limit is found to exhibit fluctuations at all length scales, with algebraically decaying correlation functions. The well-studied case of dimer tilings furnishes an important example of such a critical system, for which the partition function [1] and several kinds of correlation functions [2] can be computed rigorously on various lattices. Other tiling problems which have been studied recently (using a variety of techniques) and found to be critical include the double dimer model [3], triangular trimers on the triangular lattice [4], and straight trimers on the square lattice [5].

In many of these cases it is known how to transform the tiling problem into an equivalent height model. Depending on the precise tile and lattice symmetries, the height may be scalar or vector valued. In the continuum limit, the height component(s) become bosonic fields for which a conformally invariant field theory can be written. Critical exponents can then be derived by the Coulomb gas technique [6], for instance by transforming special defect tiles into various types of dislocations in the height variable. It should be noted that the conformal invariance of dimer tilings has been proven rigorously [7].

The conformal field theory (CFT) of these tiling problems often takes the simple form of a collection of N free bosons, one for each height component, and the corresponding central charge is simply $c = N$. In some cases, such as dimer tilings with an interaction that tends to align neighbouring tiles [8], it is known how to introduce interactions that make the critical exponents vary continuously, but keep c constant.

This behaviour should be contrasted with that of lattice models of self-avoiding loops. Indeed, loop models are ubiquitous in two-dimensional statistical physics, since they arise as exact graphical expansions of the partition functions of Ising and Potts models [10], the O(n) vector model [11], and percolation—each case corresponding to a precise choice of the lattice and the microscopic realization of the loops. In the partition function, each closed loop is assigned a weight n. When $-2 < n \leqslant 2$, these models can again be treated using Coulomb gas techniques, but the non-locality of n makes up for a CFT which is richer than that of simple tiling models. In particular, c now depends continuously on n, and the CFT is a *deformed* free field theory of the Liouville type (with a background electric charge) [12].

In this note we consider another tiling problem (T-shaped tetrominoes on the square lattice, see figure 1) which—for a certain choice of the local tile weights—turns out to be combinatorially equivalent to a loop model, and so in particular has a continuum limit described by a Liouville CFT. Thus, interestingly, the particular shape of the tiles conveys exactly the non-local information that is needed to define the loops. Meanwhile, the oriented loops act simply as the level lines of the corresponding (scalar) height field. The partition function of the loop model is in turn equal to that of the (bond-inhomogeneous) Q-state Potts model—with Q being related to the tile weights—which is also known in combinatorics and graph theory as the (multivariate) Tutte polynomial.

On one hand, the equivalence presented here thus adds to the list of different representations of the Potts model, which includes (apart from the loop incarnation) the original spin representation, the random cluster model, a twisted six-vertex model, and a height model of the RSOS type (see [10] for a review). Since even for the simplest case of the square lattice, the Potts model is in general unsolved (except for certain critical choices of the temperature [10]), each new representation is potentially of interest, since it might allow the application of different techniques of resolution. In the case at hand, this remark even applies to numerical studies, due to the existence of powerful Monte Carlo techniques for sampling tiling problems [9]. On the other hand, our equivalence permits to take over known results for the (critical) Potts model to describe the continuum limit of the tiling problem.

The tile weights that we use (see below for precise definitions) consist of two different parts that we shall call a and b type. The a-type part (see figure 1) depends on the orientation of the tile in space and could be realized physically by placing the T-shaped molecules in an external electric field that couples to their multipolar moments. Since an external field may be inhomogeneous, it is natural to make the a-interaction vary throughout space. Preferring one orientation over the other will in general render the model non-critical, although more interesting situations may arise if the a-interaction is periodic (staggered) or random.

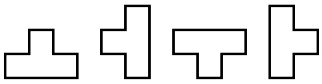

Figure 1. Weights of the four types of T-shaped tetrominoes.

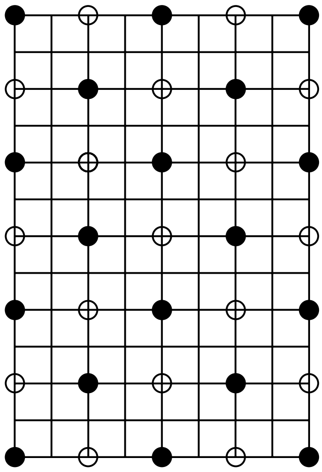

Figure 2. The domain $\mathcal{D}$ to be tiled (here with $m = 2$, $n = 3$). Some of the vertices are distinguished by black or white circles.

Meanwhile, the b-type part (see figure 3) is more subtle and describes the interaction of the tiles with a periodic crystal field which can be realized by the adsorbing substrate. This latter part is directly related to the variable Q in the Potts model.

Our work generalizes a recent result of Korn and Pak [13] who considered the same tiling problem, but did not introduce local weights of the tiles. Some of the equivalences that we use are inspired by those used by these authors, but the basic idea of decorating the tiles by oriented arcs (see figure 4) is new.

2. T-tetromino tilings

We consider tiling an $M \times N$ rectangle of the square lattice (henceforth denoted $\mathcal{D}$) with T-shaped tetrominoes, i.e., tiles of size four lattice faces in the shape of the letter T. The four different orientations of the tiles are shown in figure 1. Note that the boundary of each tile comprises ten vertices, of which eight are corners. Figure 2 shows the domain $\mathcal{D}$ to be tiled. The circles distinguish two subsets of the vertices which we shall henceforth refer to as either black (filled circles) or white (empty circles).

We have the following remarkable theorem.

Theorem 1 (Walkup [14])**.** *$\mathcal{D}$ is tileable iff $M = 4m$ and $N = 4n$. In a valid tiling, no tile corner covers a white vertex, whereas black vertices are covered by tile corners only.*

Theorem 1 implies that tiles can be distinguished not only by their orientation (cf figure 1) but also by their position relative to the white vertices. There are two possible situations, as

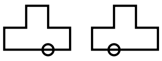

Figure 3. Weights depending on the relative position of a white vertex within a tile.

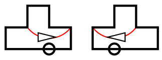

Figure 4. Two ways of decorating a tile by an oriented half-arch, depending on its position relative to a white vertex.

shown in figure 3, depending on which of the two cornerless vertices of the tile covers the white vertex. (This applies to any of the four orientations of the tiles, although figure 3 only illustrates the first orientation.)

Note that the lattice B of black vertices is a tilted square lattice of edge length $2\sqrt{2}$. This can be divided into two sublattices, B_{even} and B_{odd}, which are both straight square lattices of edge length 4. Our convention is that all black vertices on the boundary of $\mathcal{D}$ belong to B_{odd}.

We similarly divide the lattice W of white vertices into two sublattices W_{even} and W_{odd}. Referring again to figure 2, we adopt the convention that white vertices on the lower and upper boundaries of $\mathcal{D}$ belong to W_{odd} and that those on the left and right boundaries belong to W_{even}.

Equivalently, any vertex in B or W belongs to the odd (resp. even) sublattice whenever its distance from the bottom of $\mathcal{D}$ is twice an even (resp. odd) integer.

Finally, we define the generating function (partition function)

$$F_{\mathcal{D}}(\{a\},\{b\}) = \sum_{T\in\mathcal{T}(\mathcal{D})} \prod_{t\in T} a_{o(t)}^{w(t)} \prod_{j=1}^{2} b_j^{B_j}, \tag{1}$$

where $\mathcal{T}(\mathcal{D})$ is the set of all T-tetromino tilings of $\mathcal{D}$, and t is a single tile within the tiling T. The weights $a_{o(t)}^{w(t)}$ depend on the orientation $o(t) = 1, 2, 3, 4$ of the tile t (cf figure 1) and on the (unique) white vertex $w(t)$ that it covers. Moreover, B_j is the number of tiles of position $j = 1, 2$ relative to the white vertices (cf figure 3). Note that the weights b_j are not vertex dependent.

Evidently, the tiles covering the white vertices of the boundary $\mathcal{B}$ of the domain $\mathcal{D}$ have their orientation fixed (their long side is parallel to $\mathcal{B}$). It follows that those boundary tiles contribute the same a-type weight to any tiling. Without loss of generality, we can therefore henceforth set $a_{o(t)}^{w(t)} = 1$ for any t such that $w(t) \in \mathcal{B}$.

It is convenient to illustrate the distinction of figure 3 in another way, by decorating the interior of each tile by an oriented half-arch (see figure 4). We then have the following.

Lemma 1. *In any valid tiling, the orientation of the half-arches is conserved across the junctions of the tiles.*

Proof. By rotational and reflectional symmetry, it suffices to consider the junction between a tile in the a_1-type position (cf figure 1) and the neighbouring tile immediately North–East of it.

Consider then the situation where the first tile is of the b_1 type (cf figure 4). The position of the neighbouring tile is constrained by theorem 1 to have the correct location of its white

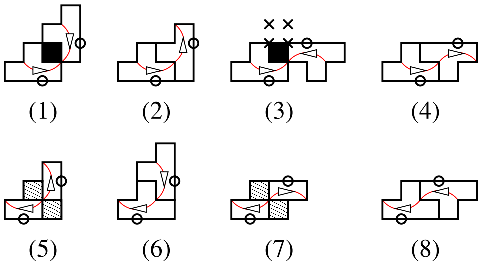

Figure 5. Arrow conservation at tile junctions.

vertex. When the neighbouring tile has its long side vertical, there are just two possibilities, shown in figure 5, panels (1) and (2). The possibility (1) is disallowed since it leaves an untiled hole (shown in black), whereas (2) is allowed. When the neighbour tile's long side is horizontal, we have the possibilities (3) and (4). But (3) is actually disallowed, since, when adding a further tile to cover the lattice face shown in black, a white point would be placed in one of the four vertices marked by a cross, and neither of these vertices is a valid position for a white point according to theorem 1. In summary, only the possibilities (2) and (4) are allowed, and both of those conserve the arrow orientation across the tile junction.

Consider next the situation where the first tile is of the b_2 type. A neighbour tile with a vertical long side leads to the possibilities (5) and (6), but (5) is disallowed since it leads to two overlaps (shown hatched). Similarly, when the neighbour's long side is horizontal, one can rule out (7) due to two overlaps, whereas (8) is allowed. In summary, only (6) and (8) are allowed and compatible with arrow conservation. □

Definition 1. *Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set E. The multivariate Tutte polynomial of G with parameter Q and edge weights $\mathbf{v} = \{v_e\}_{e\in E}$ is*

$$Z_G(Q, \mathbf{v}) = \sum_{A\subseteq E} Q^{k(A)} \prod_{e\in A} v_e, \tag{2}$$

where $k(A)$ denotes the number of connected components in the subgraph (V, A).

For the relation of the multivariate Tutte polynomial to the more standard two-parameter Tutte polynomial $T_G(x, y)$ we refer to section 2.5 of [15]. In the physics literature, $Z_G(Q, \mathbf{v})$ is better known as the partition function of the Q-state Potts model in the Fortuin–Kasteleyn formulation [16].

In the remainder of the paper we let $V = B_{\mathrm{even}}$ and E the associated natural set of horizontal and vertical edges of length 4. In other words, G is an $(m-1) \times (n-1)$ rectangle G, as referred to in the abstract. Furthermore, let $w(e)$ be the natural bijection between edges $e \in E$ and white vertices $w \in W\backslash\mathcal{B}$ that are *not* on the boundary $\mathcal{B}$ of $\mathcal{D}$.

We can now state our main result which is

Theorem 2. *The generating function $F_{\mathcal{D}}(\{a\}, \{b\})$ of T-tetromino tilings of the $4m \times 4n$ rectangle $\mathcal{D}$ depends only on the parameters $\{a\}$ through the combinations a_1a_3 and a_2a_4. For the specific choice of tile weights*

$$a_1a_3 = \begin{cases} x_e & \text{for} \quad w(e) \in W_{\mathrm{even}}\backslash\mathcal{B} \\ 1 & \text{otherwise} \end{cases} \qquad a_2a_4 = \begin{cases} x_e & \text{for} \quad w(e) \in W_{\mathrm{odd}}\backslash\mathcal{B} \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

and

$$b_1 = (b_2)^{-1} = q^{1/4} \tag{4}$$

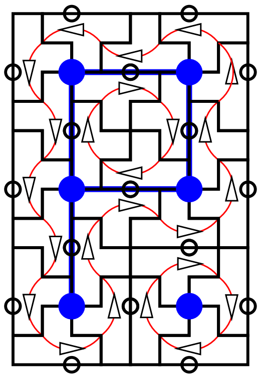

Figure 6. A valid tetromino tiling of the domain $\mathcal{D}$ shown in figure 1. The vertices V entering the definition of the Tutte polynomial (2) are shown as fat solid circles, and the edges of the subset $A \subseteq E$ are shown as fat lines.

we have the identity

$$Q^{mn/2} F_{\mathcal{D}}(\{a\}, \{b\}) = Z_G(Q; \mathbf{v}) \tag{5}$$

with the correspondence between parameters

$$Q = (q + q^{-1})^2, \qquad v_e = (q + q^{-1}) x_e \qquad \text{for} \quad e \in E. \tag{6}$$

Proof. By theorem 1, any white vertex $w \in W \backslash \mathcal{B}$ is at the junction between the long sides of exactly two tiles. If the long side is horizontal, the contribution of the a-type weight is $a_1^w a_3^w$, and if it is vertical the contribution is $a_2^w a_4^w$. This proves the first part of the theorem. (Above we have already explained that one may set all $a_i^w = 1$ when $w \in W \cap \mathcal{B}$.)

By lemma 1 the oriented half-arches form continuous curves with a consistent orientation (see figure 6 for an example). Moreover, by the consistency of the position of the white vertices (cf figures 2 and 3), these curves cannot end on $\mathcal{B}$. Therefore, the half-arches form a set of oriented cycles.

Let C be the set of all possible configurations of cycles arising from valid tilings of $\mathcal{D}$, but disregarding the orientations of the cycles. The summation in the generating function (1) can then be written $\sum_{\mathcal{T}(\mathcal{D})} = \sum_C \sum_{\mathcal{T}(\mathcal{D})|C}$, where the last sum is over cycle orientations only. A close inspection of panels (2), (4), (6) and (8) in figure 5 reveals that the orientation of a given oriented cycle c_0 can be changed independently of all other oriented cycles in the tiling $T \in \mathcal{T}(\mathcal{D})$ by shifting the tiles traversed by c_0 by one lattice unit to the left, right, up or down, but without moving any tiles not traversed by c_0. In particular, this transformation does not change the orientation of any tile (cf figure 1), and so does not change the a-type weights. We have therefore

$$F_{\mathcal{D}}(\{a\}, \{b\}) = \sum_C \prod a_{o(t)}^{w(t)} \sum_{\mathcal{T}(\mathcal{D})|C} \prod_{j=1}^{2} b_j^{B_j}. \tag{7}$$

(The remainder of the proof parallels a construction used in [17] to show the equivalence between the Potts and six-vertex model partition functions.)

The partial summation $\sum_{\mathcal{T}(\mathcal{D})|C} \prod_{j=1}^{2} b_j^{B_j}$ appearing in (7) can be carried out by noting that the b-type weights (4) simply amount to weighting each turn of c_0 through an angle $\pm\pi/2$ by the factor $q^{\pm 1/4}$. Since the complete cycle turns a total angle of $\pm 2\pi$ depending on its orientation, the product of the b-type weights associated with the cycle c_0, summed over its two possible orientations, yields the weight $q + q^{-1} = Q^{1/2}$. Denoting by $\ell(C)$ the number of cycles in C, we have therefore found that

$$\sum_{\mathcal{T}(\mathcal{D})|C} \prod_{j=1}^{2} b_j^{B_j} = Q^{\ell(c)/2}. \tag{8}$$

It now remains to sum over the un-oriented cycles C. There is an easy bijection between C and the edge subsets $A \subseteq E$ appearing in (2). Namely, an edge $e \in E$ belongs to A iff it is not cut by any cycle in C (see figure 6). Conversely, given A, the cycles are constructed so that they cut no edge in A, cut all edges in $E \backslash A$, and are reflected off the boundary $\mathcal{B}$ in the vertices $W \cap \mathcal{B}$. Under this construction, the a-type weights (3) simply give $\prod_{e\in A} x_e$. Collecting the pieces this amounts to

$$F_{\mathcal{D}}(\{a\}, \{b\}) = \sum_{A\subseteq E} Q^{\ell(A)/2} \prod_{e\in A} \frac{v_e}{Q^{1/2}}. \tag{9}$$

Using finally the Euler relation, and remarking that $|V| = mn$, one obtains (5). □

Let us discuss a couple of special cases of theorem 2. First, when there are no a-type weights (i.e., setting $a_{o(t)}^{w(t)} = 1$ for any $o(t)$ and $w(t)$), the Tutte polynomial has $v_e = q + q^{-1}$ for any $e \in E$. The tiling entropy defined by

$$S_G(q) \equiv \lim_{m,n\to\infty} (F_{\mathcal{D}})^{\frac{1}{mn}} \tag{10}$$

is then related to that found for the Tutte polynomial (*alias* Potts model) by Baxter [18] using the Bethe Ansatz method. For $0 < Q < 4$ one sets $q \equiv \mathrm{e}^{\mathrm{i}\mu}$ with $\mu \in (0, \pi/2)$ and finds

$$\log S_G(q) = \int_{-\infty}^{\infty} \mathrm{d}t \frac{\sinh[(\pi - \mu)t]\tanh(\mu t)}{t \sinh(\pi t)}, \tag{11}$$

whereas for $Q > 4$ one sets $q \equiv \mathrm{e}^{\lambda}$ and finds

$$\log S_G(q) = \lambda + 2\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-n\lambda}\tanh(n\lambda)}{n}. \tag{12}$$

Finally, for $Q = 4$ one has

$$S_G(1) = \left(\frac{\Gamma(5/4)}{\Gamma(3/4)}\right)^4. \tag{13}$$

An even more special case of the identity (5) arises when $b_1 = b_2 = 1$ as well. $F_{\mathcal{D}}$ is then the unweighted sum over all T-tetromino tilings of $\mathcal{D}$. In terms of the standard Tutte polynomial $T_G(x, y)$ one then has $F_{\mathcal{D}} = 2T_G(3, 3)$, as first proved by Korn and Pak [13].

References

[1] Kasteleyn P W 1961 *Physica* **27** 1209
Temperley H N V and Fisher M E 1961 *Phil. Mag.* **6** 1061
[2] Fisher M E and Stephenson J 1963 *Phys. Rev.* **132** 1411
[3] Dubédat J 2003 *Elect. Commun. Probab.* **8** 28–42
[4] Verberkmoes A and Nienhuis B 1999 *Phys. Rev. Lett.* **83** 3986–9
[5] Ghosh A, Dhar D and Jacobsen J L 2006 *Phys. Rev.* E (*Preprint* cond-mat/0609322) (at press)
[6] Nienhuis B 1987 Coulomb gas formulation of two-dimensional phase transitions *Phase Transitions and Critical Phenomena* vol 11 (New York: Academic) pp 1–53
[7] Kenyon R 2001 *Ann. Probab.* **29** 1128

[8] Alet F, Jacobsen J L, Misguich G, Pasquier V, Mila F and Troyer M 2005 *Phys. Rev. Lett.* **94** 235702
Alet F, Ikhlef Y, Jacobsen J L, Misguich G and Pasquier V 2006 *Phys. Rev.* E **74** 041124
[9] Krauth W and Moessner R 2003 *Phys. Rev.* B **67** 064503
[10] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[11] Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062–5
[12] Kondev J 1997 *Phys. Rev. Lett.* **78** 4320–3
[13] Korn M and Pak I 2004 *Theor. Comput. Sci.* **319** 3–27
[14] Walkup D W 1965 *Am. Math. Mon.* **72** 986–8
[15] Sokal A D 2005 *Surveys in Combinatorics, 2005* ed Bridget S Webb (Cambridge: Cambridge University Press) pp 173–226 (*Preprint* math.CO/0503607)
[16] Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 536–64
[17] Baxter R J, Kelland S B and Wu F Y 1976 *J. Phys. A: Math. Gen.* **9** 397–406
[18] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445–8